Symmetric Traveling Salesman

Polytope and problem

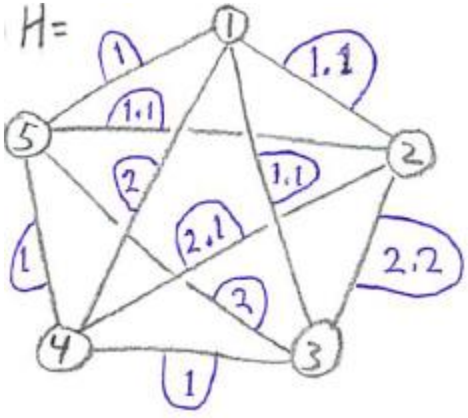
1) Problem

- Given: a set of cities and the symmetric distances (or costs) between each pair.
- A tour visits each city once and returns to the start.
- Find the tour with minimal total distance (cost).
- We can safely assume that the distances form a metric, obeying the triangle inequality.

2) Model and 3) Example

- The *n* cities and distances are modeled as a complete graph K_n with weighted edges.
- A tour is a Hamiltonian cycle *c*, and its total distance *p* is the sum of its weights.

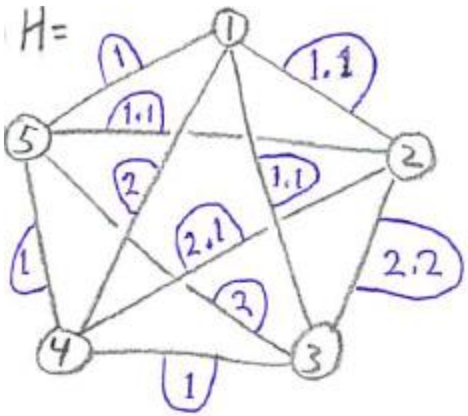
Example *n*=5:



An example tour is *c*=1,3,4,2,5,1. It's total distance is *p* = 6.3.

4) Polytope

- Our vertices $\boldsymbol{x}(c)$ correspond to Hamiltonian cycles c of the graph K_n. They have $\binom{n}{2}$ components, one for each edge.
- The component for edge $\{i,j\}$ is $x_{ij}=1$ if the edge is in the cycle; $x_{ij}=0$ if not.
- Our objective function $p=\boldsymbol{d}\cdot\boldsymbol{x}(c)$ outputs total distance. The vector $\boldsymbol{d}$ has components the given edge weights.



5) Example

$\boldsymbol{d}$ = <1.1, 1.1, 2, 1, 2.2, 2.1, 1.1, 1, 2, 1>

Vertices:

c	$\boldsymbol{x}(c)$	$p = \boldsymbol{d} \cdot \boldsymbol{x}(c)$
1,2,3,4,5,1	<1,0,0,1,1,0,0,1,0,1>	p = 6.3
1,2,3,5,4,1	<1,0,1,0,1,0,0,0,1,1>	p = 8.3
1,2,4,3,5,1		
1,2,4,5,3,1		
1,2,5,3,4,1		
1,2,5,4,3,1		
1,3,2,4,5,1		
1,3,2,5,4,1		
1,4,2,3,5,1		
1,4,2,5,3,1		
1,5,2,3,4,1		
1,5,2,4,3,1		

Etc….

6) Known Facts

Dimensions:
0, 2, 5, 9, 14 ... *n(n-3)/2*

- Numbers of vertices in *nth* polytope:

1, 3, 12, 60, ... *(n-1)!/2*

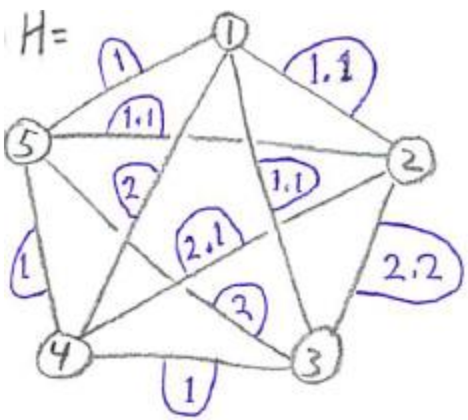
- Numbers of Facets:

0, 3, 20 ,100, 3437, 194187, 42104442, …

Source:
https://www.zib.de/groetschel/pubnew/paper/groetschelpadberg1979a.pdf

7) Greedy Algorithm

- Pick a starting city; choose the edge with least weight; repeat (don’t return to any city twice until the last step.)



The greedy algorithm starting at node 1
gives the cycle *c* =1,5,4,3,2,1 with total distance *p*=6.3.

8) LP using http://www.zweigmedia.com/RealWorld/simplex.html

Minimize $p = 1.1x_{12} + 1.1x_{13} + 2x_{14} + 1x_{15} + 2.2x_{23} + 2.1x_{24} + 1.1x_{25} + 1x_{34} + 2x_{35} + 1x_{45}$ subject to

$x_{12} + x_{13} + x_{14} + x_{15} = 2$
$x_{12} + x_{23} + x_{24} + x_{25} = 2$
$x_{23} + x_{13} + x_{34} + x_{35} = 2$
$x_{24} + x_{34} + x_{14} + x_{45} = 2$
$x_{15} + x_{25} + x_{35} + x_{45} = 2$
$x_{12} >= 0$
$x_{13} >= 0$
$x_{14} >= 0$
$x_{15} >= 0$
$x_{23} >= 0$
$x_{24} >= 0$
$x_{25} >= 0$
$x_{34} >= 0$
$x_{35} >= 0$
$x_{45} >= 0$
$x_{12} <= 1$
$x_{13} <= 1$
$x_{14} <= 1$
$x_{15} <= 1$
$x_{23} <= 1$
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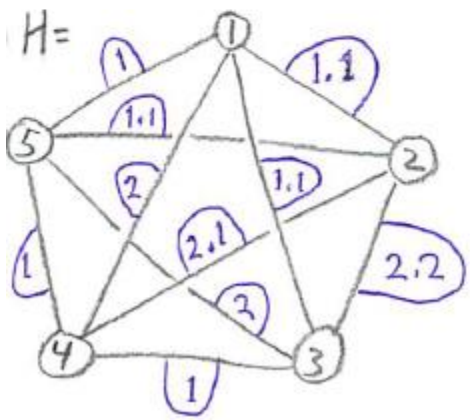
$x_{34} <= 1$
$x_{35} <= 1$
$x_{45} <= 1$



Optimal Solution: p = **5.3**; $x_{12} = 1$, $x_{13} = 1$, $x_{14} = 0$, $x_{15} = 0$, $x_{23} = 0$, $x_{24} = 0$, $x_{25} = 1$, $x_{34} = 1$, $x_{35} = 0$, $x_{45} = 1$
[That's the vertex vector ***x**(c)* = <1,1,0,0,0,0,1,1,0,1> which is the cycle *c*=1,2,5,4,3,1]
[Note: You can cut and paste the above into the link given at the top and see the steps of the simplex method with the Dantzig rule.]