The Permutahedron

a) The face poset

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- Example for $n = 5$:

$$(\{4\},\{2\},\{1,3\},\{5\}) < (\{2,4\},\{1,3\},\{5\}) < (\{2,4\},\{1,3,5\})$$

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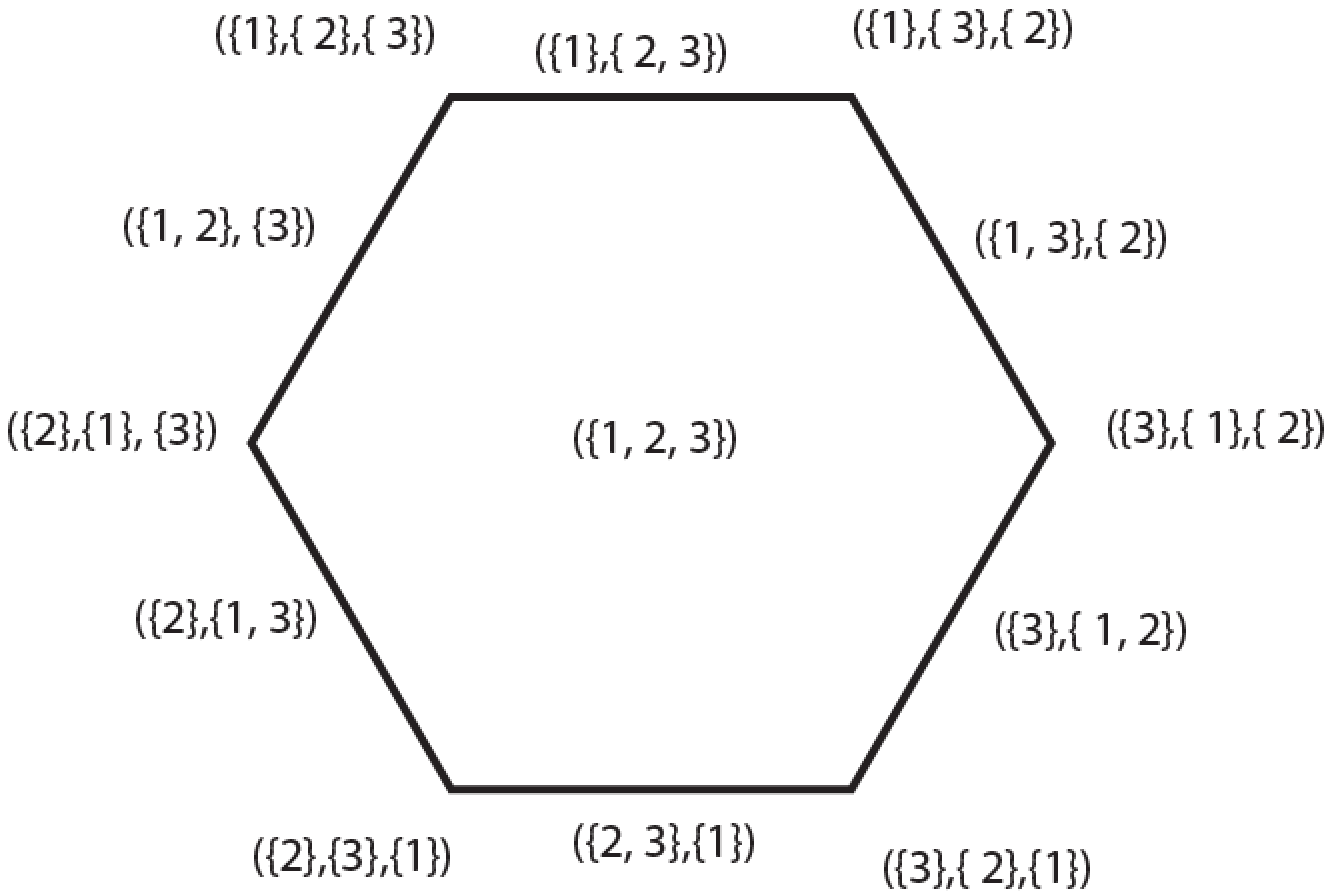
- Example for *n* = 5:

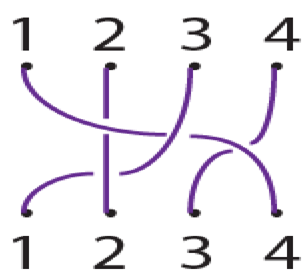
  ({4},{2},{1,3},{5}) < ({2,4},{1,3},{5}) < ({2,4},{1,3,5})
- The vertices of the polytope are the least items, which are the finest partitions, such as ({4},{2},{3},{1},{5}). These correspond to permutations.

b) Labeled permutahedron for n=3



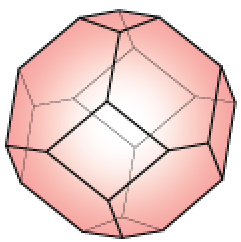
c) Permutahedron with vertex values

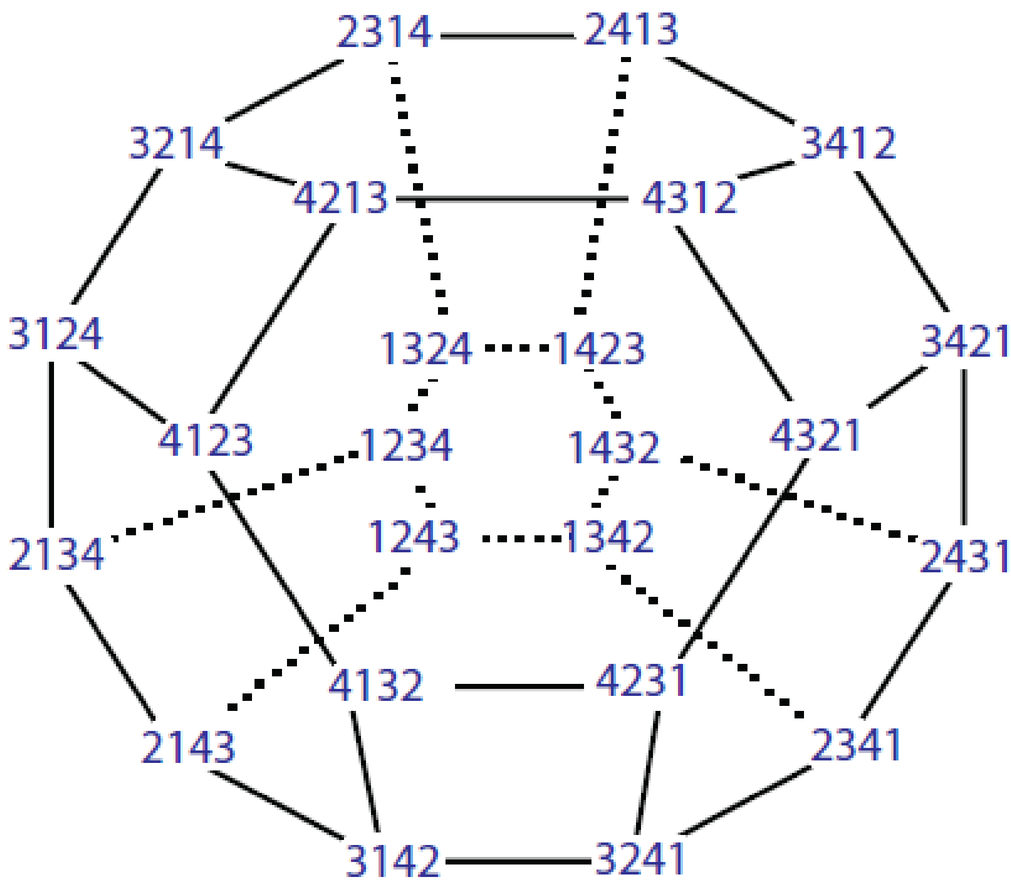
- Each vertex corresponds to a permutation.
- Example for n=4: ({4},{2},{1},{3}) corresponds to:



- The vertex coordinates are found by just listing the outputs of the permutation: (4,2,1,3) as a point in $\mathbf{R}^4$.

c) cont. Example in dimension 3.





d) Known facts

- Dimensions:

0, 1, 2, 3, ... *n-1*

- Numbers of vertices in *nth* polytope:

1, 2, 6, 24, 120, ... *n!* [OEIS A000142]

- Facets:

0, 2, 6, 14, 30 ... $2^n -2$ [OEIS A000918]

- f-vectors:

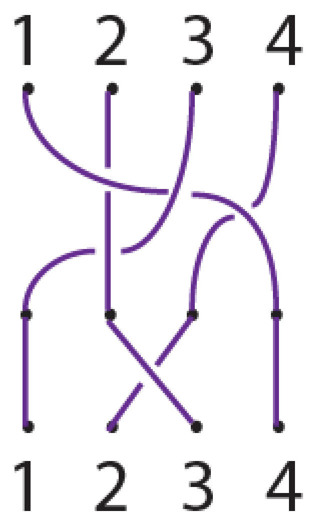
1, 2, 1, 6, 6, 1, 24, 36, 14, 1, ... [OEIS A019538]

e) Skeleton lattice

The permutations of [n] are ordered. We say a permutation is less than another if you can get to the other by composing with a series of transpositions.

Example on n=4: (4,2,1,3) < (4,3,1,2) sinc

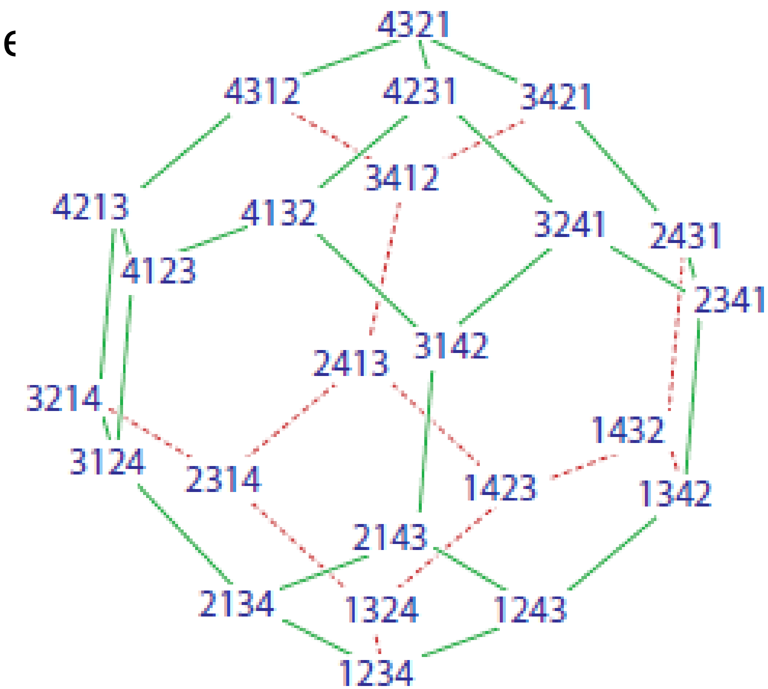
(4,2,1,3)



Transpose

(2,3)

= (4,3,1,2).



Hasse Diagram

for n=4.

f) Space tiling

- The n-dimensional permutahedron tiles n-dimensional space. In 3d, it is one of only 5 polytopes that can tile 3d space with translated (“slid over”) copies of itself.