Catalan numbers C_n

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{n!\,(n+1)!}}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

$= 1, 1, 2, 5, 14, 42, 132, 429, \ldots$

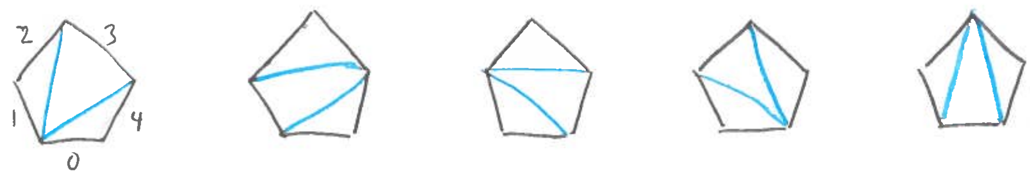
$n=0$ $\quad$ $n=3$

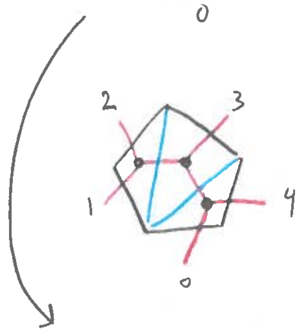
Some things they count:

$n=2$

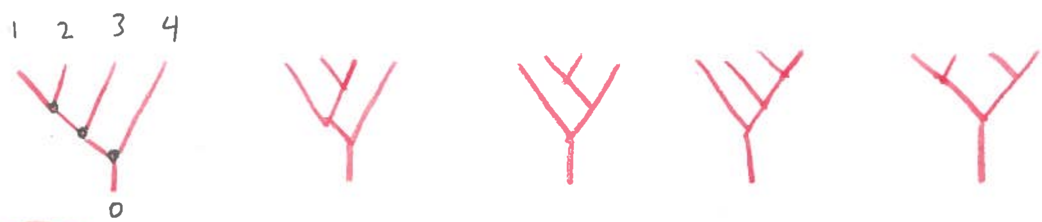
triangulations of the rooted $(n+2)$-gon

$n=3$

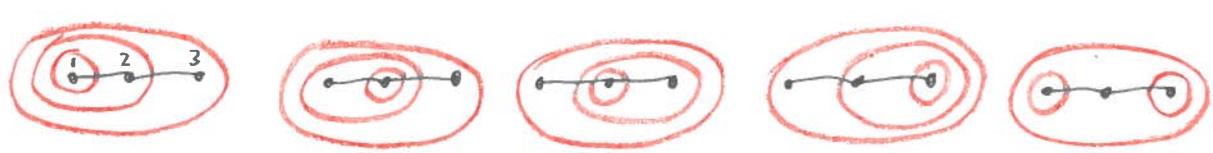




trees

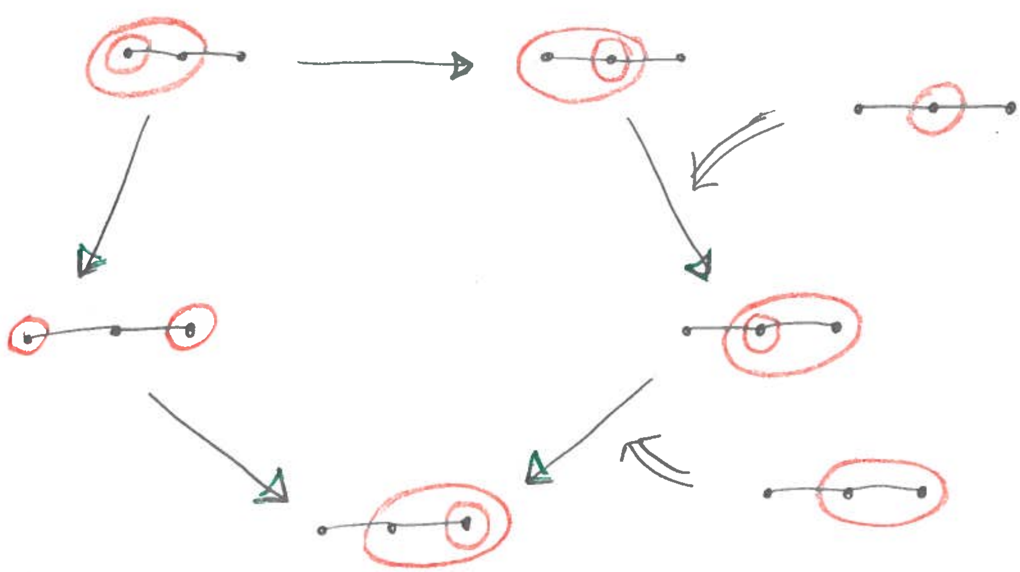


tubings

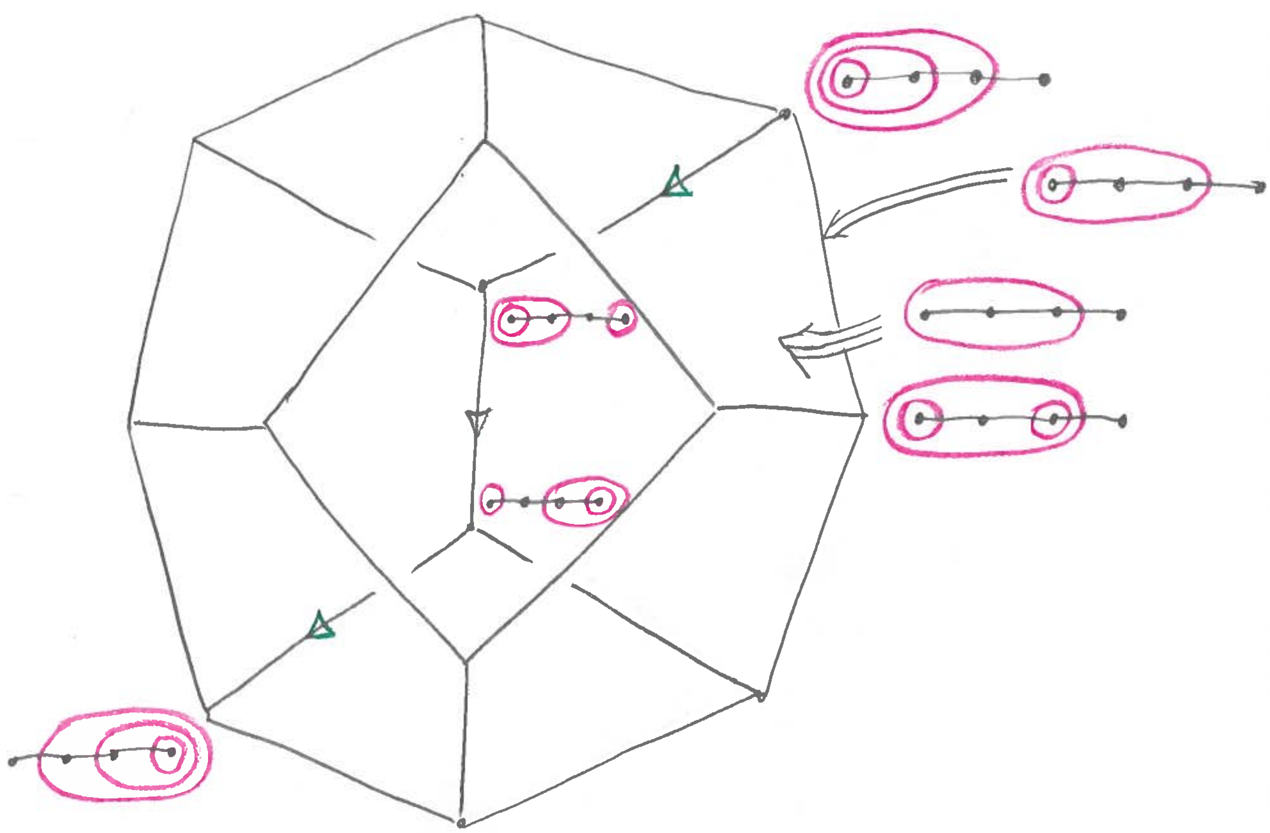


The move: pick one tube (connected induced subgraph, circled) and slide it to the right.

The rule: 2 tubes must be compatible: nested or far apart.



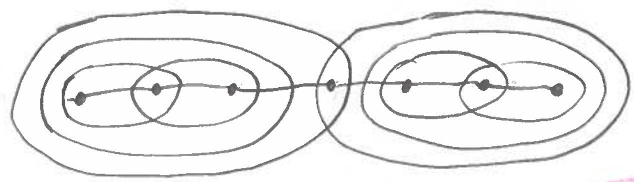
$n=4$



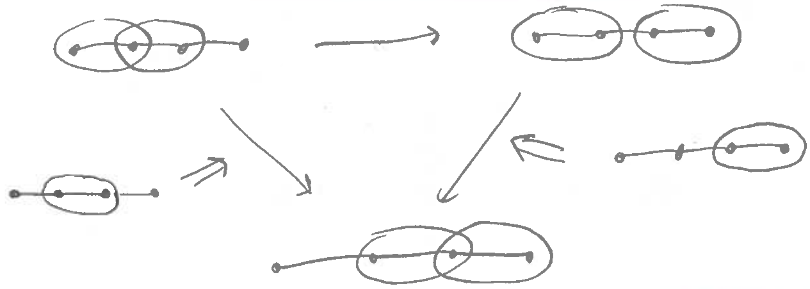
These tubings for any n always label polytopes; they are $\cong$ to the face posets of the associahedra.

Relax the Rule: two tubes can be close, or even intersect, but any 3 tubes must include a pair that is compatible.

[Since singleton tubes and tubes of one less than n nodes are compatible with everything, leave them out.]



$n=4$



$n=5$

, ...

(see next page.)

→ For n nodes, there can be at most

$$2(n-3) \text{ tubes.}$$

→ the number of maximal tubings for n nodes (vertices of the multi associahedron)

is

$$\begin{vmatrix} C_n & C_{n-1} \\ C_{n-1} & C_{n-2} \end{vmatrix} = C_n C_{n-2} - (C_{n-1})^2$$

$= 1, 1, 3, 14, 84, 594, \ldots$