OPERAD BIMODULE CHARACTERIZATION OF ENRICHMENT. V2

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1. Idea

In a recent talk at CT06 http://faculty.tnstate.edu/sforcey/ct06.htm and in a research proposal at http://faculty.tnstate.edu/sforcey/class_home/research.htm a definition of weak enrichment over a strict monoidal n-category is introduced.

We would like to generalize the definition of enrichment in a way which fits naturally into the world of weakened category theory, where multiplication and composition are unbiased and parameterized rather than being strictly binary and associative. The basic idea of classical enrichment is to allow a general binary product in a some category to reprise the role which the cartesian product of sets usually plays in describing binary composition of morphisms. This role is that of forming the domain for composition. It seems that in the new world we should expand the idea of enrichment to allow an unbiased and weakly associative product to form the domain for composition, while simultaneously allowing that composition itself to be weakened. The structure which appears to accomplish this is that of a bimodule. Broadly speaking a bimodule is an object upon which two other objects may act, from two different "directions." The actors are monoids: they are each assumed to have an associative, unital, self action of their own. That is, they each possess a multiplication which in turn must be respected by their action on the bimodule they share.

The philosophy here is that the structure of a bimodule will allow the simultaneous weakening of the forming of the domain (governed by the monoid on the left) and of the enriched composition itself (governed by the monoid on the right.) Let us attempt a definition of enrichment over the algebra of an operad in **Cat**. Thus we are given a categorical operad K, which governs the domain of our composition, the algebra V. The first question is what sort of operad should govern the composition itself. Two possibilities are immediately apparent: (1) to let the composition be governed by the associative operad t (for terminal) or (2) to let it be governed by the same operad K. Let's think about these possibilities in order. Thus first we will work out an example definition of an enriched category A over V by use of a (K, t)-bimodule C. We assume all categories to be contractible. Let $\circ$ denote the substitution product on **Cat** (for which an operad is a monoid.)

2. Example

Here are some more of the important pieces of our construction, especially the structural functors:

- I denotes the unit category.
- Let γ be the composition for K, let $\mathcal{J} : I \to K(1)$ be the unit for K,
- let $\theta : K(n) \times V^n \to V$ be the action of K on V,
- let $\lambda : K(n) \times C(j_1) \times \ldots \times C(j_n) \to C(\sum j_i)$ be the left action of K on C,
- let $\rho : C(n) \times t(j_1) \times \ldots \times t(j_n) \to C(\sum j_i)$ be the right action of t on C,
- and we let the following diagram commute.

$$\begin{array}{ccc} K \circ C \circ t & \xrightarrow{\lambda} & C \circ t \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} \\ K \circ C & \xrightarrow{\lambda} & C \end{array}$$

2.1. Definition. *Now an enriched category A over V governed by C is as follows:*

1. *Let A_0 denote the set of objects of A.*

2. *For each pair of objects $a, b \in A_0$ there is an object $A(a, b)$ in V. The full subcategory A^n of V^n has object set*

$$A^n = \{(A(a_{n-1}, a_n), \dots, A(a_1, a_2), A(a_0, a_1)) \in V^n \mid a_0, \dots, a_n \in A_0\}.$$

3. *For all n there is a composition action $\mu : C(n) \times V^n \to V$. It respects a unit $\eta : I \to C(1)$, which means that this diagram commutes:*

$$\begin{array}{ccc} I \times V & \xrightarrow{\eta} & C(1) \times V \\ \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle \mu} & \\ V & & \end{array}$$

4. *When restricted to A^n, the composition μ respects the left action λ. By respect of the left action we mean that the following diagram commutes:*

$$\begin{array}{ccc} K(n) \times C(j_1) \times \ldots \times C(j_n) \times A^{(\sum j_i)} & \xrightarrow{\mu^n} & K(n) \times V^n \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \theta} \\ C(\sum j_i) \times A^{(\sum j_i)} & \xrightarrow{\mu} & V \end{array}$$

5. *There is a trivial action of t on A^n given by $c : t(n) \times A^n \to A^1 \subset V$ where*

$$c(*, A(a_{n-1}, a_n), \dots, A(a_1, a_2), A(a_0, a_1)) = A(a_0, a_n).$$

The composition μ respects this action as in the commutativity of :

$$\begin{array}{ccc} C(n) \times t(j_1) \times \ldots \times t(j_n) \times A^{(\sum j_i)} & \xrightarrow{\rho} & C(\sum j_i) \times A^{(\sum j_i)} \\ \downarrow{\scriptstyle c^n} & & \downarrow{\scriptstyle \mu} \\ C(n) \times V^n & \xrightarrow{\mu} & V \end{array}$$

Now a couple of theorems.

2.2. Theorem. *Axioms 3 and 4 together imply that the following diagram commutes:*

$$\begin{array}{ccc} K(n) \times A^n & \xrightarrow{1\times\eta^n\times 1} & K(n) \times C(1)^n \times A^n \\ \Big\downarrow{\scriptstyle \theta} & & \Big\downarrow{\scriptstyle \lambda\times 1} \\ V & \xleftarrow[\mu]{} & C(n) \times A^n \end{array}$$

2.3. Theorem. *Axioms 3 and 5 together imply that the following diagram commutes:*

$$\begin{array}{ccc} t(n) \times A^n & \xrightarrow{\eta} & C(1) \times t(n) \times A^n \\ \Big\downarrow{\scriptstyle c} & & \Big\downarrow{\scriptstyle \rho\times 1} \\ V & \xleftarrow[\mu]{} & C(n) \times A^n \end{array}$$

The importance of the theorems is that they make clear how the bimodule enables the weak enrichment. Recall that the action θ is the multiplicative structure of V. Thus the first theorem shows how the weak enrichment at one "end" of the bimodule is just the same as multiplication of the hom-objects. This is exemplified by the inclusion of the n^{th} associahedron in the $n+1^{st}$ composihedron. Also recall that the trivial action c is the connection between A^n and A^1. Thus the second theorem shows how the weak enrichment at the other "end" of the bimodule collapses a string of composable hom-objects into their resultant.

The left structure of the bimodule reflects the weak composition in the base V and the right structure encodes the structure of the enriched composition. Using a contractible bimodule allows interpolation between the two.

Now lets try to generalize our definition of an enriched category A over V by using a (K,P)-bimodule C.

3. Example

Here are the new important pieces of our construction, especially the structural functors:

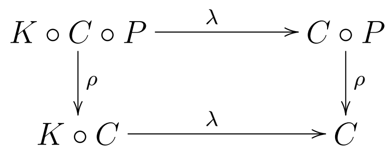
- I denotes the unit category.

- let $\theta : K(n) \times V^n \to V$ be the action of K on V,

- let $\lambda : K(n) \times C(j_1) \times \ldots \times C(j_n) \to C(\sum j_i)$ be the left action of K on C,

- let $\rho : C(n) \times P(j_1) \times \ldots \times P(j_n) \to C(\sum j_i)$ be the right action of P on C,

- and (with some doubt as to the necessity of this provision) we let the following diagram commute.



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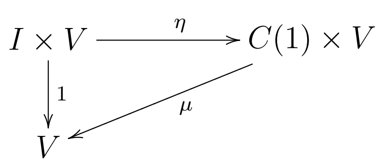
3. *For all n there is a composition action $\mu : C(n) \times V^n \to V$. It respects a unit $\eta : I \to C(1)$, which means that this diagram commutes:*



4. *When restricted to A^n, the composition μ respects the left action λ. By respect of the left action we mean that the following diagram commutes:*

$$\begin{array}{ccc} K(n) \times C(j_1) \times \ldots \times C(j_n) \times A^{(\sum j_i)} & \xrightarrow{\mu^n} & K(n) \times V^n \\ \lambda \downarrow & & \downarrow \theta \\ C(\sum j_i) \times A^{(\sum j_i)} & \xrightarrow{\mu} & V \end{array}$$

5. *The unit of C is such that for all $n > 0$ and $a_0, \ldots, a_n \in A_0$, $f \in P(n)$, we have that the composite given by*

$$P(n) \times A^n \xrightarrow{\eta} C(1) \times P(n) \times A^n \xrightarrow{\rho} C(n) \times A^n \xrightarrow{\mu} V$$

takes $(f, A(a_{n-1}, a_n), \ldots, A(a_0, a_1)) \longmapsto A(a_0, a_n)$.

3.2. Definition. *Let B be another category enriched over V governed by C. An enriched functor $F : A \to B$ is:*

1. *a function $F : A_0 \to B_0$*
2. *for each pair $a, a' \in A_0$ a morphism in V: $F_{aa'} : A(a, a') \to B(Fa, Fa')$.*
3. *...such that the following diagram commutes:*

$$\begin{array}{ccc} C(n) \times A^n & \xrightarrow{\mu} & V \\ \downarrow 1\times F^n & & \downarrow \mu(1, F^n) \\ C(n) \times B^n & \xrightarrow[\mu]{} & V. \end{array}$$

4. Comments and variations

Notice that the real job of completely composing a string of hom-objects is done by certain elements of $C(n)$ picked out by the unit as shown in axiom 5. Notice also that axioms 3 and 4 for the category over V together imply the commutativity of

$$\begin{array}{ccc} K(n) \times A^n & \xrightarrow{1\times\eta^n\times 1} & K(n) \times C(1)^n \times A^n \\ \downarrow \theta & & \downarrow \lambda\times 1 \\ V & \xleftarrow[\mu]{} & C(n) \times A^n \end{array}$$

Thus we see that at one end of the spectrum the bimodule composition action simply repeats the action of the operad, while at the other end it performs a complete composition. These are connected in various ways since the bimodule elements are contractible.

Although the axioms for an enriched category only use the action of C on A^n, to describe an arbitrary functor it seems that we need the action to be defined on all of V so as to have available what that action does to a morphism $A^n \to B^n$.

If $P = K$ there might be a way to more efficiently describe these ideas. Let $\circ$ denote the substitution product (for which an operad is a monoid.) Let K-**Mod**-K denote the monoidal category of K bimodules with product $\circ_K$, where $B \circ_K C$ is the coequalizer of the left and right actions applied to $B \circ K \circ C$. K itself is the unit for this monoidal structure. First, we might let C be a monoid with respect to $\circ_K$. The multiplication of C would be inserted somewhere?

Then the unit of C as a monoid in this framework would be a morphism $J : K \to C$. By identifying J with the composite

$$K(n) \xrightarrow{\eta} K(n) \times C(1)^n \xrightarrow{\lambda} C(n)$$

we can reduce axioms 3 and 4 to saying that the composition respects the unit J.

The other variation of note will make this construction applicable to enrichment over weak n-categories X which are defined using an operad action but not quite as operad algebras. Following the well known definition of enriching over a bicategory, we introduce a function $g : A_0 \to X_0$ and then assign to each pair $a, b \in A_0$ an object of $X(g(a), g(b))$. The rest of the definition follows accordingly.

The next thing to do is to expand a list of examples. Besides the sort of example in http://faculty.tnstate.edu/sforcey/ct06.htm (a weak homomorphism from a topological group to a loop space), we want examples of categories enriched over Trimble n-categories, or over Batanin's n-categories. Both utilize operad actions, so both should be amenable to this approach.