Operads of cellular automata and little n-cubes

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Abstract

We demonstrate how to organize 1-dimensional cellular automata into an operad of spaces. The nth term $\mathcal{C}(k)$ is the space of radius $r = k - 1$ automata. The operad composition operation involves both automata composition and shifting of domain. Pointwise operations such as addition of automata become important when we look at the structure of the individual terms in the operad, the spaces of automata with a given radius. Having adopted the discrete topology on such a space, we demonstrate an action of the little n-cubes operad on $(n-1)$-dimensional radius r automata. There are clear applications of this action to parallel programming issues. Finally we discuss ways of generalizing both the idea of an operad of automata and the n-cubes action to higher dimensional automata, using higher dimensional operads.

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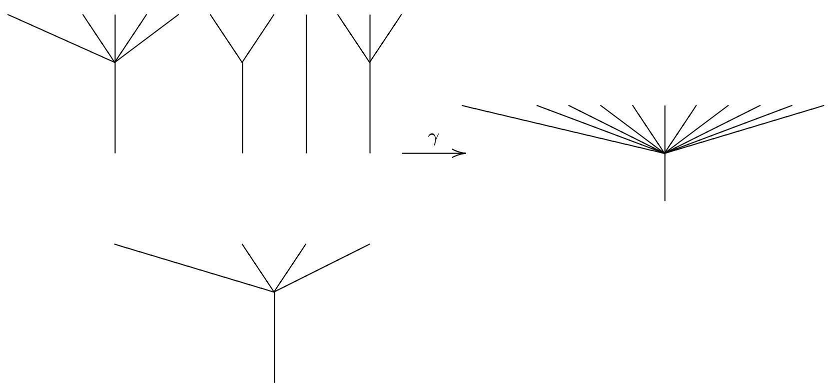
Contents

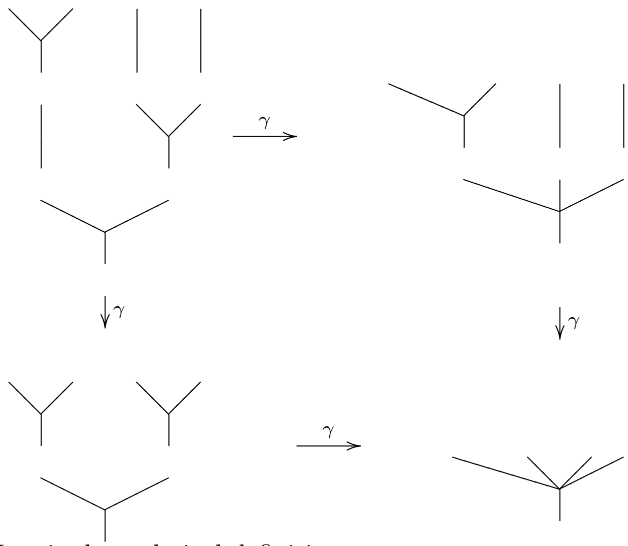
1 Introduction

Let $\mathcal{V}$ be a symmetric monoidal category. For our purposes $\mathcal{V}$ will be either the category of sets and set functions or the category of topological spaces and continuous maps. In both cases the commuting monoidal structure symbolized by $\otimes$ is provided by the cartesian product, or cross product, and the unit object I is the single point.

The two principle components of an operad are a collection, historically a sequence, of objects in a monoidal category and a family of composition maps. Operads are often described as paramaterizations of n-ary operations. Peter May's original definition of operad in a symmetric (or braided) monoidal category [4] has a composition γ that takes the tensor product of the nth object (n-ary operation) and n others (of various arity) to a resultant that sums the arities of those others. The nth object or n-ary operation is often pictured as a tree with n leaves, and the composition appears like this:



By requiring this composition to be associative we mean that it obeys this sort of pictured commuting diagram:



Here is the technical definition.

Definition 1 *An operad $\mathcal{C}$ in $\mathcal{V}$ consists of objects $\mathcal{C}(j)$, $j \geq 1$, a unit map $\mathcal{J} : I \to \mathcal{C}(1)$, and composition maps in $\mathcal{V}$*

$$\gamma : \mathcal{C}(k) \otimes (\mathcal{C}(j_1) \otimes \ldots \otimes \mathcal{C}(j_k)) \to \mathcal{C}(j)$$

for $k \geq 1$, $j_s \geq 0$ for $s = 1 \ldots k$ and $\sum\limits_{s=1}^{k} j_s = j$. The composition maps obey the following axioms

1. *Associativity: The following diagram is required to commute for all $k \geq 1$, $j_s \geq 0$ and $i_t \geq 0$, and where $\sum\limits_{s=1}^{k} j_s = j$ and $\sum\limits_{t=1}^{j} i_t = i$. Let $g_s = \sum\limits_{u=1}^{s} j_u$ and let $h_s = \sum\limits_{u=1+g_{s-1}}^{g_s} i_u$.*

$$\begin{tikzcd}
\mathcal{C}(k) \otimes \left(\bigotimes\limits_{s=1}^{k} \mathcal{C}(j_s) \right) \otimes \left(\bigotimes\limits_{t=1}^{j} \mathcal{C}(i_t) \right) \arrow[r, "\gamma \otimes \mathrm{id}"] \arrow[dd, "symmetry"'] & \mathcal{C}(j) \otimes \left(\bigotimes\limits_{t=1}^{j} \mathcal{C}(i_t) \right) \arrow[d, "\gamma"] \\
 & \mathcal{C}(i) \\
\mathcal{C}(k) \otimes \left(\bigotimes\limits_{s=1}^{k} \mathcal{C}(j_s) \otimes \left(\bigotimes\limits_{u=1}^{j_s} \mathcal{C}(i_{u+g_{s-1}}) \right) \right) \arrow[r, "\mathrm{id} \otimes (\otimes^k \gamma)"'] & \mathcal{C}(k) \otimes \left(\bigotimes\limits_{s=1}^{k} \mathcal{C}(h_s) \right) \arrow[u, "\gamma"']
\end{tikzcd}$$

2. Respect of units is required. The following unit diagrams commute.

$$\begin{tikzcd}
\mathcal{C}(k)\otimes(\otimes^k I) \arrow[r, equal] \arrow[d, "1\otimes(\otimes^k \mathcal{J})"'] & \mathcal{C}(k) \\
\mathcal{C}(k)\otimes(\otimes^k \mathcal{C}(1)) \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
I\otimes\mathcal{C}(k) \arrow[r, equal] \arrow[d, "\mathcal{J}\otimes 1"'] & \mathcal{C}(k) \\
\mathcal{C}(1)\otimes\mathcal{C}(k) \arrow[ur, "\gamma"] &
\end{tikzcd}$$

2 An operad of cellular automata

Consider 1 dimensional K-valued cellular automata where K is a set of values. There is an operad $\mathcal{C}$ of sets (or discrete topological spaces) where $\mathcal{C}(k)$ is the space of radius $r = k-1$ (neighborhood $2r+1$) cellular automata. Thus $\mathcal{C}(1)$ is the space of radius 0 automata, and the unit map $\mathcal{J} : \{1\} \to \mathcal{C}(1)$ simply chooses the identity automata. Composition involves both automata composition and shifting of domain. An element of $\mathcal{C}(k)$ is a function f of $2k-1$ variables. At times it will be convenient to label these as $x_1 \ldots x_{2k-1}$ but of course the function f is a cellular automata and so is meant to be interpreted an an update rule, where the ith cell has value a_i and the new value $a'_i = f(a_{i-r} \ldots a_{i+r})$. The translation is that old cell value a_i will be the variable x_{r+1}. The operad operation $\gamma : \mathcal{C}(k) \times (\mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k)) \to \mathcal{C}(j)$ consists of composing $f(x_1 \ldots x_{2k-1})$ with k other ordered functions $g_1 \ldots g_k$ by composing in the odd variables to get $f(g_1, x_2, g_2, x_4, g_3, \ldots x_{2(k-1)}, g_k)$. Since we are composing f with k other automata we will need to describe where these other automata are evaluated in order to furnish inputs for f. If g_n is in $\mathcal{C}(j_n)$ and thus has radius $r_n = j_n - 1$ and so is a function of $2r_n + 1$ variables, then the new radius of the composition of all our automata must be

$$r' = j - 1 = \sum_{n=1}^{k} j_n - 1 = \sum_{n=1}^{k} r_n + k - 1.$$

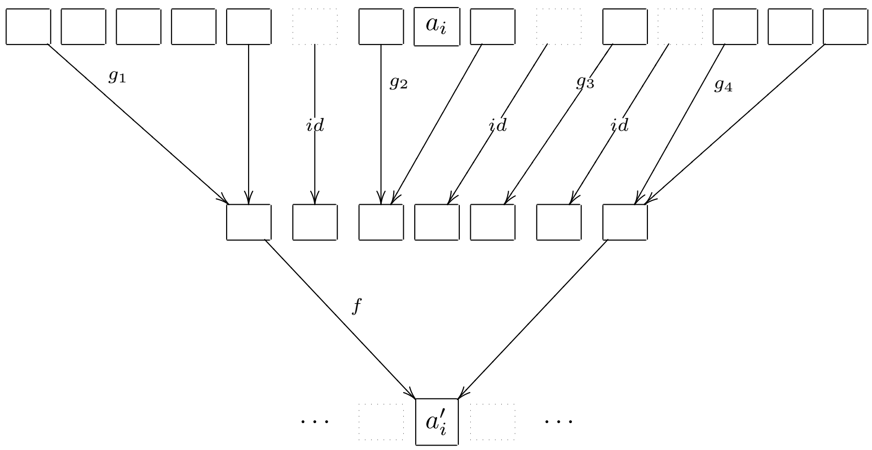
Thus for our composition to give an operad, we need to evaluate g_n on a neighborhood centered at the cell location

$$i - \left(\sum_{s=n}^{k} r_s - \sum_{s=1}^{n} r_s + k - 2n + 1 \right)$$

The even variables $x_2 \ldots x_{2(k-1)}$ of f are assigned values taken identically from the cells that lie between the new domains of the functions g_n. Specifically then x_{n+1} for n odd gets the value of the cell at location

$$i - \left(\sum_{s=n}^{k} r_s - \sum_{s=1}^{n} r_s + k - 2n - r_n \right).$$

This is all most easily seen through an example. In the following figure we are composing an element of $\mathcal{C}(4)$ (a radius 3 automata f) with 4 automata g_n, where $g_1 \in \mathcal{C}(3)$ (radius 2), $g_2 \in \mathcal{C}(2)$ (radius 1), $g_3 \in \mathcal{C}(1)$ (radius 0), and $g_4 \in \mathcal{C}(2)$ (radius 1 again.)



Theorem 1 *The sets $\mathcal{C}(j)$, the unit map $\mathcal{J}$ and the compositions γ described above form the structure of an operad.*

There is also a related family of operads where $\mathcal{C}(k)$ is radius $n(k-1)$ automata for a whole number n. Just as in the $r = k-1$ case where we composed in the variables $x_{2j+1}\ ;\ \ j = 0 \ldots k-1$ now we compose in the variables $x_{2nj+1}\ ;\ \ j = 0 \ldots k-1$.

3 n-cubes action on radius r cellular automata

Consider a generalization of $\{0, 1, \ldots k-1\}$-valued cellular automata to $[0, k)$-valued cellular automata. Addition in this context will always be mod k. There is a nice way of allowing little $(n+1)$-cubes to act on the space of $[0, k)$-valued n-dimensional radius r cellular automata. Here is an example of the action for $n = 1$. Picture a unit square centered at $(0, \frac{1}{2})$ with a smaller included square centered at the location (x, y). Let w be the (horizontal) width of the smaller square and h be its (vertical) height. Let $d = [kx]$ and $v = k(y - \frac{h}{2})$. This arrangement will act upon a rule $a_i' = f$ with radius r, so a function of $2r+1$ variables. These latter we label a_{i+j} for $j = -r \ldots r$. The dimensions and location of the small square modify the input variables a_{i+j} as follows:

$$a^*_{i+j} = h(a_{i+d+j\frac{2[wr]+1}{2r+1}}) + v$$

The new rule is $a_i' = f(a^*_{i+j})$. In effect the width and horizontal displacement of the included square shrink and shift the domain of f. The height and vertical position of the square shrink and shift the inputs of f. If there are m small squares acting on m rules $f_1 \dots f_m$ then the new rule is

$$a'_i = f_1(a^1_{i+j}) + \ldots + f_m(a^m_{i+j})$$

where the superscript on the input variable indicates modification by the respective small square.

Theorem 2 *The action of the little 2-cubes on the space of radius r $[0,k)$-valued cellular automata described above make that space into an operad algebra of the 2-cubes operad.*

With care this can be specialized to the $\{0, 1, \dots k-1\}$-valued case.

4 Higher dimensions and concurrent programming

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