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Nested nestings and Acyclonesto-cosmohedra

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Abstract. Acyclonestohedra are generalisations of Stasheff associahedra and graph associahedra defined on the data of a partially ordered set or, more generally, an acyclic realisable matroid on a building set. Recently it has been shown that associahedra admit non-simple truncation into cosmohedra that encode the flat-space cosmological wavefunction coefficients of tr(ϕ^3) theory. We show the acyclonestohedra also admit a non-simple truncation: into acyclonesto-cosmohedra, also called poset cosmohedra in the poset case. Each face of the poset cosmohedron is labelled by a nested nesting of the poset. This extended abstract describes the resulting combinatorics; the full paper has more to say about physical motivation. As part of the proof sketch, we demonstrate here that acyclonesto-cosmohedra can be obtained as sections of graph cosmohedra.

Keywords: acyclonestohedron, positive geometries, polytope, cosmohedron, oriented matroid

1 Introduction and Summary

The *acyclonestohedra* [11] provide a large class of polytopes whose faces are products of polytopes in the same class. This family includes, as special cases, the classical associahedron which describes the scattering amplitudes of biadjoint ϕ^3 theory [1], as well as the graph associahedra of [5] that appear in cosmological contexts [2]. They also encompass the poset associahedra of [8], which have not yet found application to physical processes.

We generalise the construction of graph cosmohedra in [9] to define acyclonesto-cosmohedra; these further generalise the classical cosmohedron in [2]. The key realisation is that for any polytope with faces indexed by nested sets, the nested sets themselves come equipped with a Hasse diagram which can be further imbued with its own nesting. These ideas are advertised in Figure 1. We provide evidence that acyclonesto-cosmohedra can be obtained as sections of graph cosmohedra, this generalises similar observations made for the acyclonestohedra in [11].

This paper is organised as follows. In section 2, we review the definition of acyclonestohedra. In section 3, we associate generalisations of the cosmohedron to acyclonestohedra and present realisations and examples thereof.

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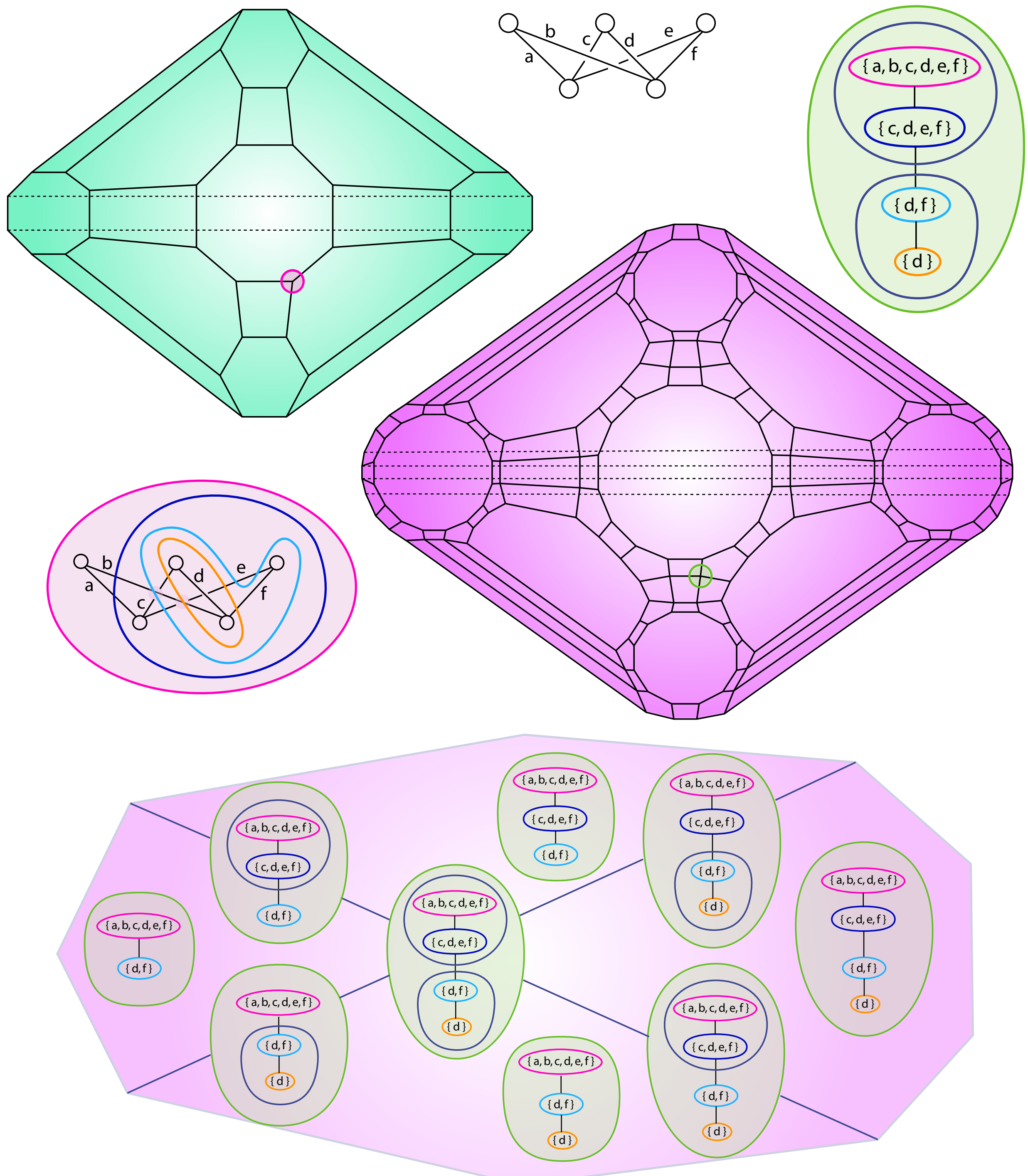

Figure 1: The acyclonestohedron (top-left) and associated acyclonesto-cosmohedron (mid-right) for nestings of the poset $K_{2,3}$ (center top). The acyclonestohedron is shown as a realisation in [13], and both polytopes as realisations here in Figure 3. One vertex of each polytope is circled, with the corresponding maximal nesting τ of $K_{2,3}$ (center left) and maximal nested nesting $(\tau, \mathcal{N})$ top-right, and below, zoomed in to see face inclusion.

2 Acyclonestohedra and their realisation

We begin by defining building sets and nestings, which generalise the notion of tubes and tubings on a graph. In general, the terms nesting, nested set, tubing, and piping are closely related: the first two are synonymous and the the second two are specializations to graphs and posets respectively.

Definition 1 ([12, 6, 7])**.** A *building set* $\mathcal{B}$ on a ground set S is a collection of nonempty subsets of S such that

- for any $s \in S$, then $\{s\} \in \mathcal{B}$;
- whenever $B, B' \in \mathcal{B}$ with $B \cap B' \neq \emptyset$, then $B \cup B' \in \mathcal{B}$.

A *connected component* of a building set $(S, \mathcal{B})$ is an inclusion-maximal element of $\mathcal{B}$; the set of connected components is denoted by $\max(\mathcal{B}) \subseteq \mathcal{B}$. A *nesting* $\mathcal{N}$ of a building set $\mathcal{B}$ is a subset $\max(\mathcal{B}) \subseteq \mathcal{N} \subseteq \mathcal{B}$ such that

- whenever $B, B' \in \mathcal{N}$, then either $B \subseteq B'$ $B' \subseteq B$ or $B \cap B' = \emptyset$;
- for any finite collection of pairwise disjoint elements $B_1, \ldots, B_k \in \mathcal{N}$ (with $k > 1$), then $B_1 \cup \cdots \cup B_k \notin \mathcal{B}$.

The collection of nestings $\{\mathcal{N} \setminus \max(\mathcal{B}) \mid \mathcal{N} \text{ is nesting}\}$ under reverse inclusion define the poset of faces of a convex polytope called the *nestohedron*. The facets of the nestohedron are labelled by nestings of the form $\{B\}$ for $B \in \mathcal{B}$. These facets factorise into products of two nestohedra defined on the *restriction* and *contraction* of $\mathcal{B}$ to $\{B\}$. Where, for any subset $R \subseteq S$, the *restriction* $\mathcal{B}_{|R}$ and *contraction* $\mathcal{B}_{/R}$ of $\mathcal{B}$ to S are defined as the building sets

$$\mathcal{B}_{|R} := \{B \in \mathcal{B} \mid B \subseteq R\}, \qquad \mathcal{B}_{/R} := \{B \setminus R \mid R \not\supseteq B \in \mathcal{B}\}. \tag{2.1}$$

Definition 2. A *signed set* $X = (X, \sigma$ is a $\mathbb{Z}_2$-graded set, i.e. a set X together with a an assignment of signs $\sigma\colon X \to \{+1, -1\}$ to every element. We may formally write such a set as $X = X^+ - X^- = x_1 + x_2 + \cdots - y_1 - y_2 - \cdots$ where $x_1, x_2, \ldots \in X^+$ are the elements with degree $+1$ and $y_1, y_2, \ldots \in X^-$ are the elements with degree -1; thus $-X = X^- - X^+$ is the signed set with all degrees reversed. An *oriented matroid* $(S, \mathcal{C})$ on a finite set S is a collection of signed sets (called *signed circuits*) $\mathcal{C}$ such that

- $\emptyset \notin \mathcal{C}$
- if $C \in \mathcal{C}$, then $-C \in \mathcal{C}$
- if $X \in \mathcal{C} \ni Y$, and $X^+ \cup X^- = Y^+ \cup Y^-$, then $X = Y$ or $X = -Y$
- if $X, Y \in \mathcal{C}$ with $X \neq -Y$ and $s \in X^+ \cap Y^-$, then there exists a $Z \in \mathcal{C}$ such that $Z^\pm \subset (X^\pm \cup Y^\pm) \setminus \{e\}$.

Given a subset $R \subseteq S$, the *restriction* $(S,\mathcal{C})_{|R}$ and *contraction* $(S,\mathcal{C})_{/R}$ are the oriented matroids given by

$$(S,\mathcal{C})_{|R} := (R, \{C \in \mathcal{C} \mid C^+ \cup C^- \subseteq R\}), \tag{2.2}$$
$$(S,\mathcal{C})_{/R} := (S \setminus R, \{(C^+ \setminus R) - (C^- \setminus R) \mid C \in \mathcal{C}\}) \tag{2.3}$$

respectively. An oriented matroid is *acyclic* if it does not have a signed circuit whose elements are all positive.

Definition 3 ([11]). An *oriented building set* $(S,\mathcal{B},\mathcal{C})$ is a building set $(S,\mathcal{B})$ together with an oriented matroid $(S,\mathcal{C})$ on the same ground set S. An *acyclic nesting* of an oriented building set $(S,\mathcal{B},\mathcal{C})$ is a nesting $\mathcal{N} \subset \mathcal{B}$ of $(S,\mathcal{B})$ such that, for every $B \in \mathcal{N}$, the oriented matroid $\big((S,\mathcal{C})_{|B}\big)_{/\bigcup\{N \in \mathcal{N} \mid N \subsetneq B\}}$ is acyclic (the notation $\bigcup$ means the union of all elements of a collection of sets, and $\subsetneq$ denotes proper subset.) When $(S,\mathcal{C})$ is realisable, the collection $\{\mathcal{N} \setminus \max\mathcal{B} \mid \mathcal{N} \text{ is an acyclic nesting}\}$ under reverse inclusion is the poset of faces of a convex polytope called the *acyclonestohedron* of $(S,\mathcal{B},\mathcal{C})$.

From the definition, it follows that the unique codimension 0 face (the interior of the polytope) is the unique nesting $\max\mathcal{B}$ (which is trivially acyclic), whilst the facets (codimension 1 faces) are in canonical bijection with those sets $B \in \mathcal{B}$ such that the oriented matroids $(S,\mathcal{C})_{|B}$ and $(S,\mathcal{C})_{/B}$ are both acyclic, and the vertices (maximal-codimension faces) are in canonical bijection with acyclic nestings that are maximal under inclusion.

If the oriented matroid $(S,\mathcal{C})$ is realised by the vectors $(a_i)_{i\in S}$ that span a k-dimensional vector space, the dimension of the acyclonestohedron is given by $k - |\max\mathcal{B}|$, where $|\max\mathcal{B}|$ is the number of connected components of $\mathcal{B}$. Given a realisable oriented building set $(S,\mathcal{B},\mathcal{C})$ and a facet given by $B \in \mathcal{B}$ such that $(S,\mathcal{C})_{|B}$ and $(S,\mathcal{C})_{/B}$ are both acyclic, then the facet of the acyclonestohedron corresponding to B factorises as

$$\text{facet for } B = \text{acyclonestohedron for } (S,\mathcal{B}_{|B},\mathcal{C}_{|B}) \times \text{acyclonestohedron for } (S,\mathcal{B}_{/B},\mathcal{C}_{/B}).$$

This may be applied recursively to higher-codimension faces.

Example 1 ([8]). Given a poset P, let the set of its covers be $S := \{(i,j) \in P^2 \mid i \lessdot j\}$; this is, equivalently, the set of edges of the Hasse diagram G of P. On S, we may construct the building set $(S,\mathcal{B})$ associated to the line graph $L(G)$ as well as the realisable oriented matroid $(S,\mathcal{C})$ associated to the digraph structure of G. Then the acyclonestohedron corresponding to the oriented building set $(S,\mathcal{B},\mathcal{C})$ is the Galashin poset associahedron for P. The building set can be described as the *pipes*, or connected convex subposets, and the nestings, or *pipings*, are collections of pipes that are pairwise nested or disjoint, and for which any subset, if collapsed, will produce an acyclic collapse of the Hasse diagram.

2.1 ABHY-like realisations of acyclonestohedra

The mathematics literature [13, 11] contains realisations of acyclonestohedra in terms of intersections of half-spaces that generalise the ABHY-like realisations of graph associahedra given in [9].

Suppose that we are given an oriented building set $(S, \mathcal{B}, C)$ and that C is realised by a collection of vectors $a_i \in V^*$ that span a finite-dimensional real vector space V^*. For each $B \in \mathcal{B}$ such that $(S, C)_{|B}$ and $(S, C)_{/B}$ are both acyclic, define the kinematic variable $X_B \colon V \to \mathbb{R}$ as the affine function

$$X_B = \sum_{i \in B} a_i - \sum_{\substack{B' \in \mathcal{B} \\ B' \subseteq B}} c_{B'}, \tag{2.4}$$

where the c_B are nonnegative real numbers (the *cut parameters*) for each $B \in \mathcal{B}$, chosen such that $c_B > 0$ is a positive real number whenever B contains more than one element and $c_B = 0$ whenever B contains only one element. In particular, we have a cut parameter c_κ for each connected component $\kappa \in \max \mathcal{B}$. Note that the X_B are not linearly independent if the vectors a_i realising the oriented matroid (S, C) are not linearly independent; one has the relations

$$\sum_{i \in B} \lambda_i \left(X_B - \sum_{\substack{B' \in \mathcal{B} \\ B' \subseteq B}} c_{B'} \right) = 0 \text{ whenever } \sum_{i \in B} \lambda_i a_i = 0. \tag{2.5}$$

Then the ABHY-like realisation of the acyclonestohedron is given by the set of points $v \in V$ such that

$$\begin{aligned} &X_B(v) \geq 0 \text{ for every } B \in \mathcal{B} \text{ such that } (S, C)_{|B} \text{ and } (S, C)_{/B} \text{ are acyclic} \\ &X_\kappa(v) = 0 \text{ for every } \kappa \in \max \mathcal{B}. \end{aligned} \tag{2.6}$$

This manifestly generalises the ABHY-like realisation for graph associahedra given in [9]. On the other hand, when the vectors realising the oriented matroid are not all linearly independent, we must impose additional conditions on the cut parameters c_B in addition to their positivity. A sufficient condition to satisfy these exotic kinematic constraints is to impose

$$c_B \ll c_{B'} \tag{2.7}$$

whenever $|B| < |B'|$; to be precise, it suffices to have $c_{B'}/c_B \leq R$, where $R > 1$ is a certain constant depending only on $(S, \mathcal{B}, C)$ [11, Def. 2.16].

3 Acyclonesto-cosmohedra

In this section, we associate to every acyclonestohedron a non-simple polytope called the *acyclonesto-cosmohedron* that generalises the cosmohedron for Stasheff associahedra [2] and graph cosmohedra [9].

3.1 Definition of acyclonesto-cosmohedra

Intuitively, in a cosmohedron, each face of the original positive geometry is refined into a poset of faces. Since faces in the acyclonestohedron correspond to nestings, it follows that we are to associate a nesting to a nesting, that is, to construct nested nestings; the poset of such nested nestings then define the acyclonesto-cosmohedron.

More concretely, recall that, for any acyclic nesting $\tau \subseteq \mathcal{B}$ on an oriented building set $(S, \mathcal{B}, \mathcal{C})$, the elements of τ are partially ordered by inclusion. The Hasse diagram of the poset $(\tau, \subseteq)$ is a rooted forest due to the requirement of elements in τ to be pairwise nested or disjoint, with the roots given by $\max \mathcal{B}$. (Since we are dealing with forests (acyclic graphs), the orientation does not matter, and the resulting building set will be the same as the building set on the line graph $L(G_\tau)$ of the Hasse diagram G_τ of $(\tau, \subseteq)$.) This naturally leads to the following definition.

Definition 4. Given a building set $(S, \mathcal{B})$, a *nested nesting* $(\tau, \mathcal{N})$ is a nesting $\tau \subseteq \mathcal{B}$ together with a nesting $\mathcal{N} \subseteq \mathcal{P}(\{(i, j) \in \tau \times \tau \mid i \lessdot j\})$ on (the Hasse diagram of) the poset $(\tau, \subseteq)$. Nested nestings are ordered by operations of collapsing a nest that is minimal in the poset $\mathcal{N}$ (the edges are contracted and the nodes are identified, and given the label of the largest nest) or discarding a non-maximal nest of $\mathcal{N}$. That is, given two nested nestings $(\tau, \mathcal{N})$ and $(\tau', \mathcal{N}')$, then $(\tau', \mathcal{N}') \leq (\tau, \mathcal{N})$ means that $\mathcal{N}'$ is formed from $\mathcal{N}$ by repeatedly collapsing a minimal nest or discarding a non-maximal nest. Note that this implies that $\tau' \subseteq \tau$. The acyclonesto-cosmohedron for the realisable oriented building set $(S, \mathcal{B}, \mathcal{C})$ is a polytope whose poset of faces is equivalent to the poset of nested nestings on $(S, \mathcal{B}, \mathcal{C})$ (with the relation $\leq$ reversed).

An acyclic nesting τ of the acyclonestohedron may be identified with the nested nesting $(\tau, \operatorname{conn}(L(G_\tau)))$, where G_τ is the Hasse diagram of $(\tau, \subseteq)$ and $\operatorname{conn}(L(G_\tau))$ is the (collection of sets of vertices of) connected components of the line graph of G_τ (or, equivalently, the collection of sets of edges of each connected components of G_τ, ignoring one-vertex connected components).

This is superficially different from the definition based on 'regions' in previous literature [2, 9]; however, explicit computation shows that they agree. The regions associated to a nested nesting $(\tau, \mathcal{N})$ are in bijection with the elements of $\mathcal{N}$; each $N \in \mathcal{N}$ is a set consisting of pairs $(i, j) \in \tau \times \tau$ with $i \leq j$, and the region corresponding to N is then the 'union' of the formal differences $j \setminus i$. For the case of the classical cosmohedron of [2], we show the correspondence between collections of subpolygons (Russian dolls) and nested nestings in Figure 2.

An advantage of the present definition is that it generalises readily: one can consider nested nested nestings, nested^3 nestings, and so on, to obtain *iterated cosmohedra* (if such iterated nestings in fact are polytopal).

The acyclonesto-cosmohedron also satisfies a factorisation property generalising that

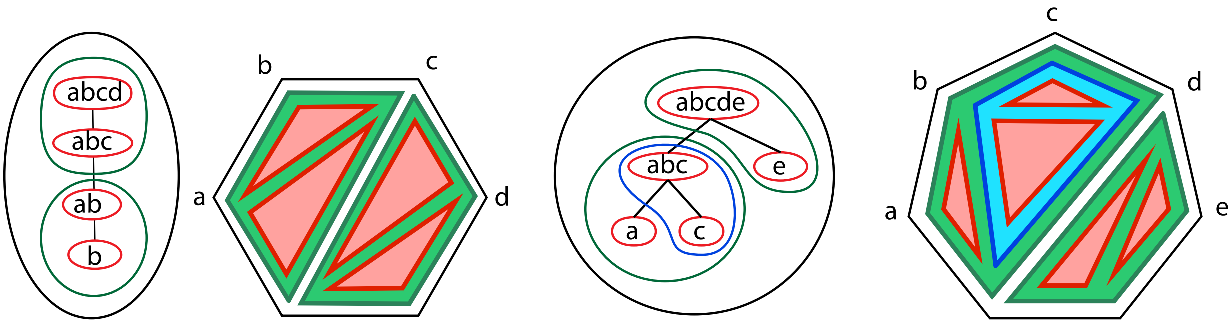

Figure 2: The maximal nested nestings here are on the building set from a path graph or totally ordered poset. They are paired with their corresponding pictures of subpolygon collections from [2].

given in [9]. A facet of the acyclonesto-cosmohedron $C_{(S,\mathcal{B},\mathcal{C})}$ for the oriented nested complex $(S,\mathcal{B},\mathcal{C})$ is given by a nesting $\tau \subseteq \mathcal{B}$, and we have the factorisation for a facet $\mathcal{F}_\tau$:

$$\mathcal{F}_\tau = A_\tau \times \prod_{B\in\tau} C_{((S,\mathcal{B},\mathcal{C})_{|B})/\bigcup\{N\in\tau \mid N\subsetneq B\}} \tag{3.1}$$

where A_τ is the poset associahedron for the poset $(\tau,\subseteq)$[1] and $C_{((S,\mathcal{B},\mathcal{C})_{|B})/\bigcup\{N\in\tau \mid N\subsetneq B\}}$ is the acyclonesto-cosmohedron associated to the oriented building set $((S,\mathcal{B},\mathcal{C})_{|B})/\bigcup\{N\in\tau \mid N\subsetneq B\}$ (this is the same restriction–contraction found in definition 3).

3.1.1 Face combinatorics and simplicity

In what follows we will assume that the building set has only one connected component. For any nested nesting $(\tau,\mathcal{N})$ the codimension of the face of the acyclonesto-cosmohedron labelled by that nested nesting is the number of nests in $\mathcal{N}$, always including the improper nest. That is, the dimension of a face is $d-|\mathcal{N}|$. Then any facet $(\tau,\mathcal{N})$ of an acyclonesto-cosmohedron has $\mathcal{N}=\{\tau\}$. We often simply draw this as the nesting τ. A facet which corresponds to a maximal nesting τ is combinatorially equivalent to the Galashin poset associahedron A_τ. A poset of nests in a nesting always has a Hasse diagram which is a tree. Thus as shown in [3] a facet equivalent to A_τ is in fact an *operahedron*, as defined in [10]. In fact, all the poset associahedra A_τ occurring in the factorisation of faces just described are operahedra.

We use the same facts to bound the degree of any vertex of the acyclonesto-cosmohedron and to generalise the fact mentioned in [2] that the cosmohedra are non-simple as polytopes.

[1] Or, equivalently, the graph associahedron for the line graph of the Hasse diagram of $(\tau,\subseteq)$; this line graph is called the *spine* in [9, (3.10)].

In the acyclonesto-cosmohedron, a vertex is a maximal nesting τ with a maximal nesting $\mathcal{N}$ of its Hasse diagram. Edges incident on that vertex are labelled via dropping a proper nest from $\mathcal{N}$ or by collapsing a minimal nest of $\mathcal{N}$. In a d-dimensional acyclonestohedron, a maximal nesting will have $d+1$ nests. Then the tree (or forest) of these tubes contains at most $\lfloor (d+1)/2 \rfloor$ minimal nests, all a single edge. This maximum occurs for instance when the Hasse diagram is linear, a totally ordered poset. (That in turn does occur if the building set is from a simple graph; it may not be the case when the building set is from a general hypergraph.) Thus for the acyclonesto-cosmohedron in this case, there is a maximum of $\lfloor (3d-1)/2 \rfloor$ edges incident to such a vertex. Thus in this case it is always a non-simple polytope for dimension $d > 2$.

The minimum degree of a vertex is of course the dimension d. That is always seen to occur for some vertices of the acyclonesto-cosmohedron, since we can find vertices $(\tau, \mathcal{N})$ where $\mathcal{N}$ is totally nested (for each tree). In this case there is only one minimal nest, and so the number of incident edges is $d-1+1 = d$.

3.2 ABHY-like realisation of acyclonesto-cosmohedra

The acyclonesto-cosmohedra can be realised in an ABHY-like fashion as intersections of half-spaces. It is shown in [11] that acyclonestohedra can be obtained as sections of graph associahedra. The same holds true for acyclonesto-cosmohedra for posets, that is, they can be obtained as sections of the graph cosmohedra associated to the line graph of the Hasse diagram of that poset.

Realization: Suppose that we are given an oriented building set $(S, \mathcal{B}, C)$ where (S, C) is realised by a collection of vectors $a_i \in V^*$ that span a finite-dimensional real vector space V^*. For each acyclic nesting $\tau \subseteq \mathcal{B}$, define the kinematic variable $Y_\tau : V \to \mathbb{R}$ as the affine function

$$Y_\tau = \sum_{B \in \tau} X_B - \sum_{B \in \tau} \delta_{B \setminus \bigcup \{N \in \tau \,|\, N \subsetneq B\}}, \tag{3.2}$$

where $\delta_{B \setminus \bigcup \{N \in \tau \,|\, N \subsetneq B\}}$ is a positive real number (additional cut parameters) associated to the subset $B \setminus \bigcup \{B' \in \tau \,|\, B' \subsetneq B\}$, and where X_B was defined in (2.4).

Then the ABHY-like realisation of the acyclonesto-cosmohedron is given by the set of points $v \in V$ such that

$$\begin{aligned} Y_\tau(v) &\geq 0 \text{ for every acyclic nesting } \tau \subseteq \mathcal{B} \\ X_\kappa(v) &= 0 \text{ for every } \kappa \in \max \mathcal{B}. \end{aligned} \tag{3.3}$$

(Of course, one also always has the additional equations (2.5) for the linear dependence amongst the X_B.) For this to realise the acyclonesto-cosmohedron, there are additional inequalities that must be satisfied by the cut parameters δ and c; it suffices to have

$$\delta_{S'} \ll \delta_{S''}, \text{ and } \delta_{S'} \ll c_B, \text{ for } |S'| < |S''| \text{ and for all } B, \text{ in addition to (2.7).} \tag{3.4}$$

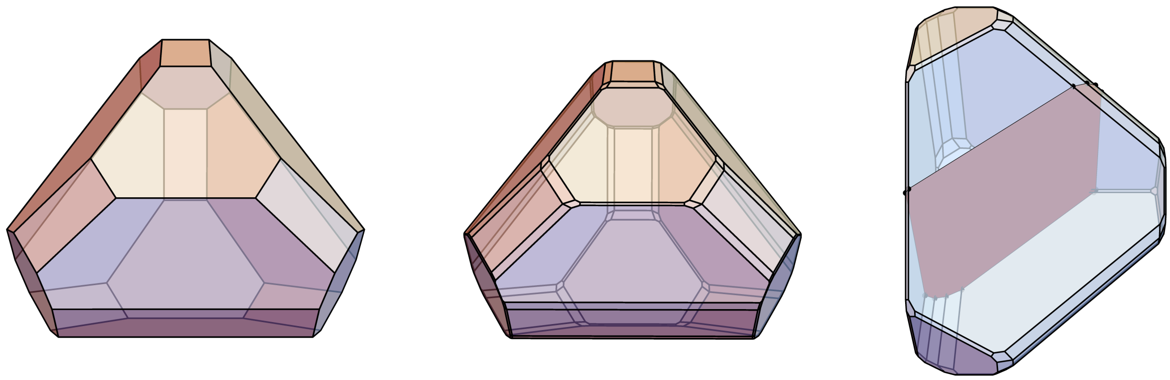

Figure 3: Explicit realisations for the acyclonestohedron (left) and its associated acyclonesto-cosmohedron (center) for the $K_{2,3}$ poset discussed in Example 4. The poset cosmohedron for the diamond poset (right) realised as a section of the graph cosmohedron for the four-cycle.

Proof sketch: The proof for the validity of our realization is straightforward for the case of poset cosmohedra, if we assume that we can begin with the realization of the graph cosmohedra as defined in [9], and follow the logic of the proof for the realization of poset associahedra in [11]. Note that the cosmohedron for a cycle-free, tree-like, poset is precisely the graph cosmohedron for the line graph of the Hasse diagram of that poset. For the acyclic restriction of a general poset we show 1) that the nested nestings of the poset cosmohedron are combinatorially equivalent as a poset to the acyclic restriction of the associated line graph cosmohedron; 2) that our realization is actually a cross section of the graph cosmohedron (an intersection with the hyperplane determined by the cycle equalities); and 3) that this cross section intersects faces of the graph cosmohedron if and only if those cells correspond to acyclic nestings.

3.3 Examples of acyclonesto-cosmohedra

The cosmohedron for the Stasheff associahedron can be seen as the acyclonesto-cosmohedron for the (totally) ordered linear poset. For the undirected simple graphs, the graph cosmohedra are described in [9].

Example 2 (Diamond poset)**.** The poset shown in Figure 4 has oriented building set:

$$S = \{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}, \qquad C = \{\pm\mathtt{a} \pm \mathtt{c} \mp \mathtt{d} \mp \mathtt{b}\}, \qquad \mathcal{B} = \mathcal{P}(S) \setminus \{\varnothing, \{\mathtt{a},\mathtt{d}\}, \{\mathtt{b},\mathtt{c}\}\}, \qquad (3.5)$$

with one trivial nesting, $\{S\}$, and 12 non-trivial nestings:

- six nestings that correspond to facets of the acyclonestohedron, of the form $\{S, B\}$ for $B \in \mathcal{B} \setminus \{\{\mathtt{a},\mathtt{c}\}, \{\mathtt{b},\mathtt{d}\}\}$ with $|B| \le 2$. (The would-be facets $\{S, \{\mathtt{a},\mathtt{c}\}\}$ and $\{S, \{\mathtt{b},\mathtt{d}\}\}$ violate the acyclicity condition.)

- six nestings that correspond to vertices of the acyclonestohedron:

$$\{S,\{\mathtt{a},\mathtt{b}\},\{\mathtt{a}\}\},\quad \{S,\{\mathtt{a},\mathtt{b}\},\{\mathtt{b}\}\},\quad \{S,\{\mathtt{c},\mathtt{d}\},\{\mathtt{c}\}\},$$
$$\{S,\{\mathtt{c},\mathtt{d}\},\{\mathtt{d}\}\},\quad \{S,\{\mathtt{a}\},\{\mathtt{d}\}\},\quad \{S,\{\mathtt{b}\},\{\mathtt{c}\}\}.$$

The Hasse diagrams for all nestings are path graphs, so we have one trivial nested nesting, $(\{S\},\varnothing)$, and the following non-trivial nested nestings:

- the six nestings that correspond to facets of the acyclonestohedron each admit a unique nested nesting. For example, to the nesting $\{S,\{\mathtt{a},\mathtt{b}\}\}$, we can associate the nested nesting $(\{S,\{\mathtt{a},\mathtt{b}\}\},\{\{(\{\mathtt{a},\mathtt{b}\},S)\}\})$.
- the six nestings that correspond to facets of the acyclonestohedron each correspond to three nested nestings, for 18 total. For example, the nesting $\{S,\{\mathtt{a}\},\{\mathtt{d}\}\}$ corresponds to the nested nestings

$$(\{S,\{\mathtt{a}\},\{\mathtt{d}\}\},\{\{(\{\mathtt{a}\},S)\},\{(\{\mathtt{a}\},S),(\{\mathtt{d}\},S)\}\}),$$
$$(\{S,\{\mathtt{a}\},\{\mathtt{d}\}\},\{\{(\{\mathtt{d}\},S)\},\{(\{\mathtt{a}\},S),(\{\mathtt{d}\},S)\}\}),$$
$$\text{and } (\{S,\{\mathtt{a}\},\{\mathtt{d}\}\},\{\{(\{\mathtt{a}\},S),(\{\mathtt{d}\},S)\}\}).$$

Therefore, the acyclonesto-cosmohedron for the diamond poset is a dodecagon. The ABHY-like realisation of the corresponding acyclonesto-cosmohedron is as follows:

$$\begin{aligned}
Y_{\{S,\{\mathtt{a},\mathtt{b}\},\{\mathtt{a}\}\}} &= X_S + 2X_{\{\mathtt{a}\}} + X_{\{\mathtt{b}\}} - c_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{a}\}} - \delta_{\{\mathtt{b}\}} - \delta_{\{\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{a},\mathtt{b}\},\{\mathtt{b}\}\}} &= X_S + X_{\{\mathtt{a}\}} + 2X_{\{\mathtt{b}\}} - c_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{a}\}} - \delta_{\{\mathtt{b}\}} - \delta_{\{\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{a},\mathtt{b}\}\}} &= X_S + X_{\{\mathtt{a}\}} + X_{\{\mathtt{b}\}} - c_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{c},\mathtt{d}\},\{\mathtt{c}\}\}} &= X_S + 2X_{\{\mathtt{c}\}} + X_{\{\mathtt{d}\}} - c_{\{\mathtt{c},\mathtt{d}\}} - \delta_{\{\mathtt{c}\}} - \delta_{\{\mathtt{d}\}} - \delta_{\{\mathtt{a},\mathtt{b}\}} \geq 0\\
Y_{\{S,\{\mathtt{c},\mathtt{d}\},\{\mathtt{d}\}\}} &= X_S + X_{\{\mathtt{c}\}} + 2X_{\{\mathtt{d}\}} - c_{\{\mathtt{c},\mathtt{d}\}} - \delta_{\{\mathtt{c}\}} - \delta_{\{\mathtt{d}\}} - \delta_{\{\mathtt{a},\mathtt{b}\}} \geq 0\\
Y_{\{S,\{\mathtt{c},\mathtt{d}\}\}} &= X_S + X_{\{\mathtt{c}\}} + X_{\{\mathtt{d}\}} - c_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{a},\mathtt{b}\}} - \delta_{\{\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{a}\},\{\mathtt{d}\}\}} &= X_S + X_{\{\mathtt{a}\}} + X_{\{\mathtt{d}\}} - \delta_{\{\mathtt{b},\mathtt{c}\}} - \delta_{\{\mathtt{a}\}} - \delta_{\{\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{b}\},\{\mathtt{c}\}\}} &= X_S + X_{\{\mathtt{b}\}} + X_{\{\mathtt{c}\}} - \delta_{\{\mathtt{a},\mathtt{d}\}} - \delta_{\{\mathtt{b}\}} - \delta_{\{\mathtt{c}\}} \geq 0\\
Y_{\{S,\{\mathtt{a}\}\}} &= X_S + X_{\{\mathtt{a}\}} - \delta_{\{\mathtt{a}\}} - \delta_{\{\mathtt{b},\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{b}\}\}} &= X_S + X_{\{\mathtt{b}\}} - \delta_{\{\mathtt{b}\}} - \delta_{\{\mathtt{a},\mathtt{c},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{c}\}\}} &= X_S + X_{\{\mathtt{c}\}} - \delta_{\{\mathtt{c}\}} - \delta_{\{\mathtt{a},\mathtt{b},\mathtt{d}\}} \geq 0\\
Y_{\{S,\{\mathtt{d}\}\}} &= X_S + X_{\{\mathtt{d}\}} - \delta_{\{\mathtt{d}\}} - \delta_{\{\mathtt{a},\mathtt{b},\mathtt{c}\}} \geq 0\\
X_S = X_{\{\mathtt{a}\}} + X_{\{\mathtt{b}\}} + X_{\{\mathtt{c}\}} + X_{\{\mathtt{d}\}} - c_{\{\mathtt{a},\mathtt{b}\}} - c_{\{\mathtt{c},\mathtt{d}\}} - c_{\{\mathtt{a},\mathtt{b},\mathtt{c},\mathtt{d}\}} &= 0\\
X_{\{\mathtt{a}\}} - X_{\{\mathtt{b}\}} + X_{\{\mathtt{c}\}} - X_{\{\mathtt{d}\}} &= 0.
\end{aligned} \tag{3.6}$$

which indeed defines a dodecagon provided that the inequalities (2.7) and (3.4) hold.

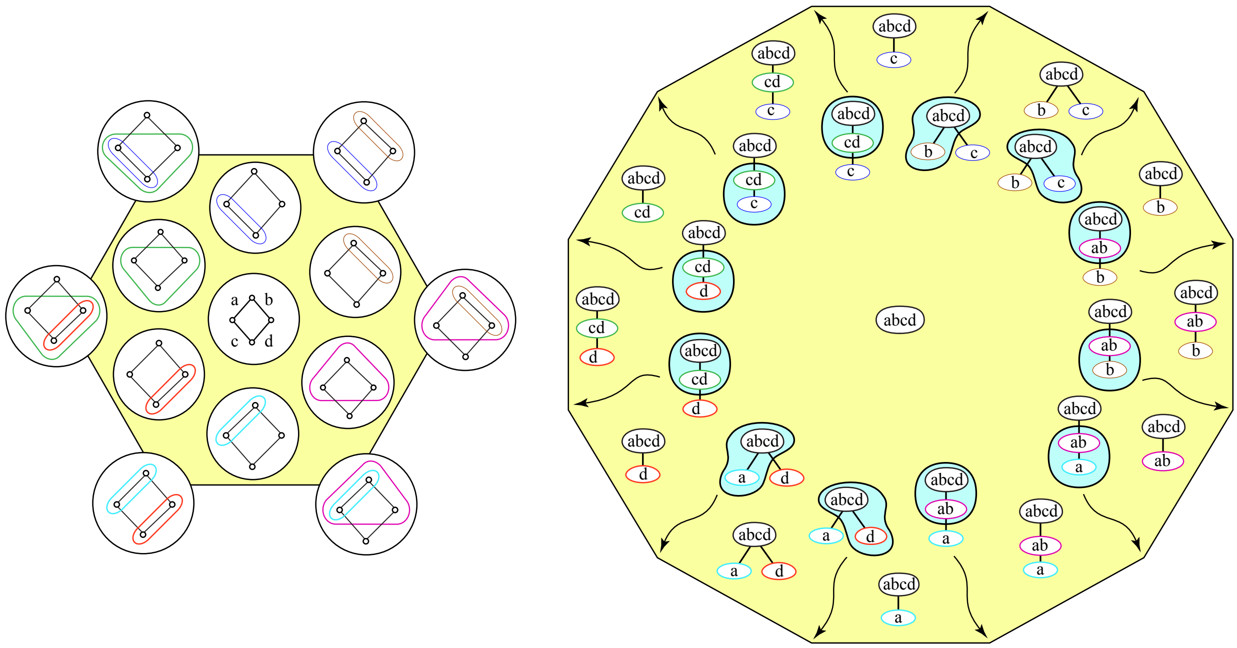

Figure 4: The poset associahedron for the diamond poset is pictured on the left, and its acyclonesto-cosmohedron is the dodecagon. This dodecagon is realized as a section of the graph cosmohedron for C_4, as illustrated in Figure 3.

Example 3 (Simplex/Permutohedron/Permutoassociahedron)**.** The building set on $S = [n]$ with a single maximal element, S, has as nestohedron the n-simplex. The Hasse diagrams of the maximal nestings are claw graphs, and nestings of those will always be totally nested. Thus the acyclonesto-cosmohedron in this case will be simple, and will recapture the combinatorics of the permutohedron, as pointed out in [9].

The poset associahedron for a claw poset ($n-1$ minimal nodes all covered by an n^{th} node) is an $(n-2)$ -dimensional permutohedron. Again every maximal nesting of the claw poset is totally nested, so the corresponding facets of the acyclonesto-cosmohedron are copies of the associahedron. Thus the acyclonesto-cosmohedron is indeed the permutoassociahedron. This example is already seen via its line graph, the complete graph, in [9].

Example 4 ($K_{2,3}$)**.** The poset with three maximal nodes, two minimal nodes, and all covering relations between them is pictured in Figure 1. Explicit realisations of both the acyclonestohedron and acyclonesto-cosmohedron for this case are shown in Figure 3. Note that the acyclonestohedron has three octagonal facets and its acyclonesto-cosmohedron has three 16-gons. As well, every maximal nesting of the poset $K_{2,3}$ is totally nested, so that the corresponding facets of the acyclonesto-cosmohedron are pentagonal: copies of the two-dimensional associahedron, which is the operahedron on the linear tree.

3.3.1 Future directions

We note that our constructions extend easily to other combinatorial polytopes that are based on a nested set paradigm, such as poset associahedra of [4], where again the Hasse diagram of each tubing is a tree. Multi-associahedra with multi-tubings of the path graph as combinatorial labels (corresponding to multidiagonalizations of polygons) can also be given a nested nesting structure. In that case the nested nesting itself would need to be acyclic, since the Hasse diagram of a multitubing can contain cycles.

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