Suppose we are given the following data:

(1) A 2-fold monoidal category $\mathcal{C}$, with tensor products $\otimes_1$ and $\otimes_2$ and interchange maps $\eta_{abcd} : (a \otimes_2 b) \otimes_1 (c \otimes_2 d) \to (a \otimes_1 c) \otimes_2 (b \otimes_2 d)$ satisfying the axioms described in [BF],
(2) A (2-fold monoidal) natural transformation Λ of the identity functor $1_{\mathcal{C}}$, and
(3) A "dimension function" $\sigma : \mathrm{Obj}(\mathcal{C}) \to \mathbb{N}$ which is *additive* over the tensor products: $\sigma(a \otimes_1 b) = \sigma(a) + \sigma(b)$ and $\sigma(a \otimes_2 b) = \sigma(a) + \sigma(b)$.

We propose the following construction of a related 2-fold monoidal category:

Definition. *The category $\mathcal{C}:\Lambda$ is constructed as follows:*

(1) *Obj$(\mathcal{C}:\Lambda)$ is the same as Obj(C).*
(2) *Hom sets in $\mathcal{C}:\Lambda$ are the same as in C, but restricted to morphisms between objects of the same dimension. That is,*

$$Hom_{\mathcal{C}:\Lambda}(a,b) = \begin{cases} Hom_{\mathcal{C}}(a,b), & \sigma(a) = \sigma(b) \\ \emptyset, & otherwise \end{cases}$$

(3) *$\mathcal{C}:\Lambda$ has the same tensor products $\otimes_1$ and $\otimes_2$ as $\mathcal{C}$.*
(4) *The interchange map for $\mathcal{C}:\Lambda$ is $(\eta:\Lambda)$ defined by*

$$(\eta:\Lambda)_{abcd} = (1_a \otimes_1 \Lambda_c^{\sigma(b)} \otimes_2 \Lambda_b^{\sigma(c)} \otimes_1 1_d) \circ \eta_{abcd}$$

where Λ_x^y indicates y-fold composition of the endomorphism $\Lambda_x : x \to x$, and by convention Λ_x^0 indicates the identity map 1_x.

Example.

Let $\mathcal{C}$ be the category of free $\mathbb{Z}$-modules with direct sum playing the role of both "products", σ equal to the rank, and the standard symmetric braiding isomorphism playing the role of the interchange map. Let Λ be multiplication by a nontrivial scalar x. The twisted interchange map from $a \oplus b \oplus c \oplus d$ to $a \oplus c \oplus b \oplus d$ can be viewed as a block matrix:

$$\begin{pmatrix} 1 & & & \\ & & x^{\sigma(b)} & \\ & x^{\sigma(c)} & & \\ & & & 1 \end{pmatrix}$$

where each entry represents a scalar of the appropriate dimension. Note that if x is not a unit then the twisted interchange is not an isomorphism.

The point, of course, is the following:

Proposition. *The category $\mathcal{C} : \Lambda$ defined above satisfies the axioms of a 2-fold monoidal category.*

Proof.

This requires nothing more than walking through the axioms in definition 1.7 of [BF], which are all routine. Remarks:

(1) Naturality of $(\eta:\Lambda)$ is straightforward but this is the reason for restricting to maps between same-dimension objects.
(2) The internal/external unit conditions are satisfied due to the fact that $\sigma(1)$ must be 0.
(3) Since we are constructing a 2-fold category there is no giant hexagon to worry about.
(4) The interesting part is the associativity constraints. The two legs of the internal associativity diagram can be reduced to

$$(1_u \otimes \Lambda_w^{\sigma(v)} \otimes \Lambda_y^{\sigma(v)+\sigma(x)} \otimes \Lambda_v^{\sigma(w)+\sigma(x)} \otimes \Lambda_x^{\sigma(y)} \otimes 1_z) \circ \eta_{uw,vx,y,z} \circ \eta_{u,v,w,x}$$

and

$$(1_u \otimes \Lambda_w^{\sigma(v)} \otimes \Lambda_y^{\sigma(v)+\sigma(x)} \otimes \Lambda_v^{\sigma(w)+\sigma(x)} \otimes \Lambda_x^{\sigma(y)} \otimes 1_z) \circ \eta_{u,v,wy,xz} \circ \eta_{w,x,y,z}$$

respectively (subscripts on the tensors are surpressed). Equality then follows by the internal associativity of the original η. This is where we need the additivity of σ over $\otimes_1$, and the fact that Λ is a monoidal natural transformation.
(5) The external associativity axiom is entirely similar and makes use of the additivity of σ over $\otimes_2$. □